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(Research Article)

Solving neutral delay differential equations using least square method based on successive integration technique

C. Kayelvizhi and A. Emimal Kanaga Pushpam *

Department of Mathematics, Bishop Heber College (Autonomous), Tiruchirappalli – 620 017, Affiliated to Bharathidasan University, Tamilnadu, India.

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Abstract

The main objective of this work is to propose the Least square method (LSM) using successive integration technique for solving Neutral delay differential equations (NDDEs). Continuous LSM and Discrete LSM have been presented by adopting different orthogonal polynomials as weighted basis functions. In this study, the most widely used classical orthogonal polynomials, namely, the Bernoulli polynomial, the Chebyshev polynomial, the Hermite polynomial, and the Fibonacci polynomial are considered. Numerical examples of linear and nonlinear NDDEs have been provided to demonstrate the efficiency and accuracy of the method. Approximate solutions obtained by the proposed method are well comparable with exact solutions. From the results it is observed that the accuracy of the numerical solutions by the proposed method increases as N (order of the polynomial) increases. The proposed method is very effective, simple, and suitable for solving the linear and nonlinear NDDEs in real-world problems.

Keywords: Least square method; Neutral delay differential equations; Orthogonal polynomials; Successive integration technique

1. Introduction

Neutral delay differential equations are a type of delay differential equations (DDEs) in which the highest-order derivative of the unknown function occurs with delay. DDEs and NDDEs arise in the fields of signal processing, digital images, control systems, epidemiology, chemical kinetics, etc. Some notable applications of DDEs and NDDEs are in electrochemical biosensor [1], cancer cells growth [2] and population model [3], human balancing models [4], quasi-static piezoelectric beams [5].

Many authors have been investigated and developed various analytical and numerical methods to solve DDEs and NDDEs. Some of them are Adams predictor corrector method [6], Homotopy perturbation method [7], Reproducing kernel Hilbert space method [8], Variational iteration method [9], Elzaki transform method [10], Haar wavelet series method [11], Higher order derivative Runge Kutta method [12], Hybrid multistep block method [13] and Generalized rational multi-step method [14] for solving DDEs and NDDEs.

The Least square method is a kind of weighted residual method to solve ordinary differential equations (ODEs). Daniele [15] has applied least square method to initial and boundary value problems of ODEs. Siti Farhana et al. [16] have solved ODEs by using LSM with an implementation of gradient method. Salisu [17] has investigated LSM for finding approximate solutions to ODEs. Parth et al. [18] have examined the performance of LSM on solving first order ODEs. Salisu and Abdulnasir [19] have used continuous LSM to solve second order ODEs.

* Corresponding author: A. Emimal Kanaga Pushpam

In this study, two kinds of LSM, namely, Continuous LSM (CLSM) and Discrete LSM (DLSM) based on successive integration technique have been presented for solving NDDEs. We adopted four different orthogonal polynomials for weighted basis functions. Numerical examples are considered for testing the efficiency of the proposed method. In section 2, basic definition of polynomials is given. The description of discrete and continuous LSM for solving NDDEs are provided in section 3. In section 4, illustrative examples are provided.

2. Basic definition of polynomials

In this study, we consider the most widely used classical orthogonal polynomials, namely, the Hermite polynomial, the Bernoulli polynomial, the Chebyshev polynomial and the Fibonacci polynomial.

2.1. Hermite Polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty,\infty)$. There are different ways to define for Hermite polynomial, one of them is the so-called Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \text{.................}(1)$$

From Eqn. (1), the recurrence relation for the polynomials can be derived as

$$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t) \qquad \text{..............................}(2)$$

$H_0(t)$ can be obtained from Eqn. (1) and the remaining terms are determined by using the recursion relation Eqn. (2). Thus, we have the following sequence of polynomials:

$$H_0(t) = 1$$

$$H_1(t) = 2t$$

$$H_2(t) = 4t^2 - 2$$

$$H_3(t) = 8t^3 - 12t$$

$$H_4(t) = 16t^4 - 48t^2 + 12$$

and so on. The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

2.2. Bernoulli Polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function of n^{th} order Bernoulli polynomial is defined by

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{xe^{xt}}{e^x - 1} \text{.....................}(3)$$

The Bernoulli polynomial is explicitly written as:

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(t^k) \text{........................}(4)$$

for n ≥ 0.

$B_0(t)$ can be obtained from Eqn. (3) and the remaining terms are determined by using the recursion relation. Thus, we have few terms of the Bernoulli polynomials as:

$$B_0(t) = 1$$

$$B_1(t) = t - 1/2$$

$$B_2(t) = t^2 - t + 1/6$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$$

$$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$$

2.3. Chebyshev Polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1, 1]$ of order n is defined as

$$T_n(\cos t) = \cos(nt) \text{ ………………….(5)}$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t) \text{………………….6)}$$

$T_0(t)$ and $T_1(t)$ can be obtained from Eqn. (5). Then the remaining terms are determined by from Eqn. (6). Thus, we have the following sequence of polynomials:

$$T_0(t) = 1$$

$$T_1(t) = t$$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 3t$$

$$T_4(t) = 8t^4 - 8t^2 + 1$$

2.4. Fibonacci Polynomial

The Fibonacci polynomials are generated by Fibonacci numbers. The recurrence relation of Fibonacci polynomial is:

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{n-1}(t) + F_{n-2}(t), & if\ n \geq 2. \end{cases}$$

Using this relation, we have the following sequence of polynomials:

$$F_0(t) = 0$$

$$F_1(t) = 1$$

$$F_2(t) = t$$

$$F_3(t) = t^2 + 1$$

$$F_4(t) = t^3 + 2t$$

3. Solving NDDEs using Least Square method based on successive integration technique

Consider the nth order NDDE

$$y^{(n)}(t) = f(t, y(t), y(t-\tau), y'(t), y'(t-\tau),), \dots, y^{(n-1)}(t), y^{(n-1)}(t-\tau),\ y^{(n)}(t-\tau)\big), t > t_0 \dots\dots\dots\dots\dots .. (7)$$

with initial conditions

$$y^{(i)}(t_0) = \emptyset(t), i = 1, 2, 3, \dots\ t \leq t_0 \text{ ………………… (8)}$$

Here τ is the delay term and $\emptyset(t)$ is the history function.

Let P(t) represent any orthogonal polynomials. For the proposed method, we assume that

$$y^{(n)}(t) \approx B^T P(t)^T = \sum_{j=0}^{N} c_j P_j(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

where N being any positive integer.

$$B^T = (c_0, c_1, \ldots c_N)$$

$$P(t) = \left(P_0(t), P_1(t) \ldots P_N(t)\right)$$

Our aim is to determine the polynomial coefficients $c_j's$. For this, we integrate Eqn. (9) with respect to t from t_0 to t,

$$\left.\begin{array}{c} y^{(n-1)}(t) = y(t_0) + \int_{t_0}^{t} B^T P_j(t)\,dt \\ y^{(n-2)}(t) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P_j(t)\,dt \\ \\ \cdots \\ \\ y'(t) = \sum_{i=0}^{n-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \ldots \int_{t_0}^{t} B^T P_j(t)\,dt \\ (n-1)\ times \\ y(t) = \sum_{i=0}^{n} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \ldots \int_{t_0}^{t} B^T P_j(t)\,dt \\ n\ times \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Now, for delay terms

$$\left.\begin{array}{c} y^{(n)}(t-\tau) = B^T P_j(t-\tau) \\ y^{(n-1)}(t-\tau) = y(t_0) + \int_{t_0}^{t} B^T P_j(t-\tau)\,dt \\ y^{(n-2)}(t-\tau) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P_j(t-\tau)\,dt \\ \\ \cdots \\ \\ y'(t-\tau) = \sum_{i=0}^{n-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \ldots \int_{t_0}^{t} B^T P_j(t-\tau)\,dt \\ (n-1)\ times \\ y(t-\tau) = \sum_{i=0}^{n} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} \ldots \int_{t_0}^{t} B^T P_j(t-\tau)\,dt \\ n\ times \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (11)$$

By substituting (10) and (11) in (7), we get the residue function R(t). The coefficients $c_j's$ can be obtained using the LSM which is based on weighted residuals minimization. In this study, we introduce Continuous Least Square Method (CLSM) and Discrete Least Square Method (DLSM).

3.1. Continuous Least Square Method

In CLSM, we make the residue function R tend to zero by minimizing the error function

$$E = \int_{\Omega} R^2(t)dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

for $t \epsilon \Omega$.

To obtain an optimum solution with minimal error E, we differentiate the Eqn. (12) with respect to c_j and then equate to zero. Thus, we have

$$\frac{\partial E}{\partial c_j} = \frac{\partial}{\partial c_j} \int R^2(t)\ dt = 0, \text{for } j = 1, 2, \dots, N,$$

which implies

$$\int_{t=0}^{t=1} R(t)\ \frac{\partial R(t)}{\partial c_j}\ dt\ = 0, \text{for } j = 1, 2, \dots, N \text{........................} (13)$$

This yields an algebraic system of linear and nonlinear equations subject to the linear and nonlinear terms involving in the Eqn. (7). By solving this system of equations, we get the respective polynomial co-efficient c_j's from which the solution of the NDDE (7) can be obtained.

3.2. Discrete Least Square Method

In DLSM, we consider the residuals at the points $t_i, 1 \leq i \leq N$. Let

$$E = \sum_{i=1}^{N} R^2(t) \quad \text{....................} (14)$$

To obtain an optimum solution with minimal error E, we differentiate the Eqn. (14) with respect to c_j and then equate to zero. Thus, we have

$$\frac{\partial E}{\partial c_j} = 0, \text{for } j = 1, 2, \dots, N,$$

This yields an algebraic system of linear and nonlinear equations. By solving this system of equations, we get the respective polynomial coefficients c_j's from which the solution of the NDDE (7) can be obtained.

4. Numerical Examples

Three examples of NDDEs have been solved by using CLSM and DLSM based on successive integration technique with four orthogonal polynomials, namely Hermite, Bernoulli, Chebyshev, and Fibonacci. Here, for convenience, in the case of CLSM, we denote them as H-CLSM, B-CLSM, C-CLSM and F-CLSM respectively. Similarly, in the case of DLSM, we denote them as H-DLSM, B-DLSM, C-DLSM and F-DLSM respectively.

4.1. Example 1

Consider the second order linear NDDE with constant delay and variable coefficient

$$ty''(t) + ty(t) + y''(t-1) + y'(t-1) = 2\cos(t-1)$$

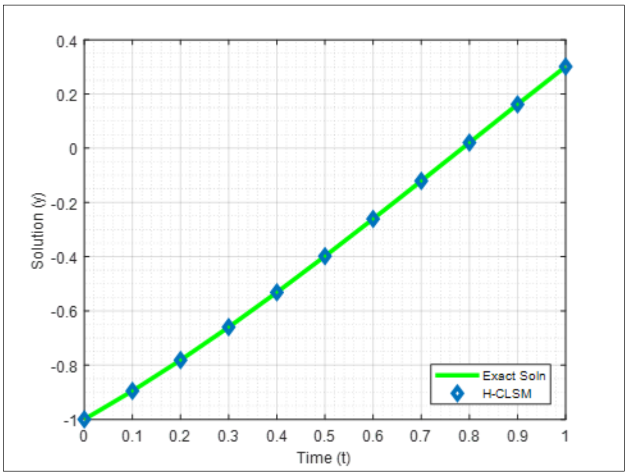
with initial condition $y(0) = -1$ and $y'(0) = 1$.

Exact solution is $y(t) = \sin(t) - \cos(t)$.

Table 1 Solutions and Absolute Error results for Example 1

t	Exact Solution	H-CLSM Solution	H-DLSM Solution	Error in H-CLSM	Error in H-DLSM
0.2	-0.78139724	-0.78139784	-0.78139783	5.999 e-07	5.876 e-07
0.4	-0.53164265	-0.53164452	-0.53164448	1.877 e-06	1.832 e-06
0.6	-0.26069314	-0.26069600	-0.26069591	2.861 e-06	2.775 e-06
0.8	0.02064938	0.02064670	0.02064682	2.673 e-06	2.554 e-06
1.0	0.30116867	0.30116777	0.30116790	8.998 e-07	7.707 e-07

The numerical solutions obtained by using the proposed methods H-CLSM and H-DLSM with N = 7 are compared with the exact solution. The results are given in Table 1. The solution graphs obtained by using the proposed methods with N = 7 are presented in Figure 1.



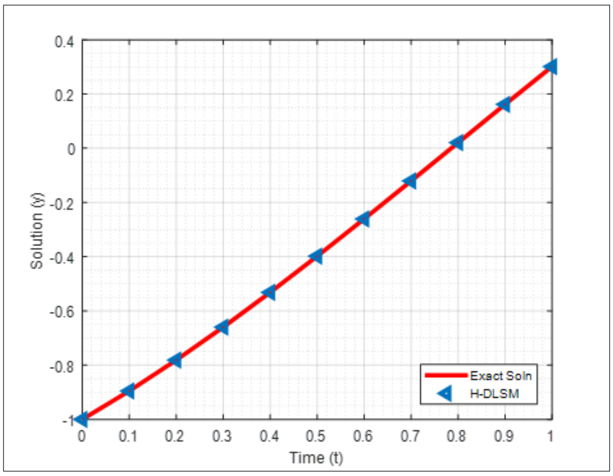


Figure 1 Solution Graphs for Example 1

4.2. Example 2

Consider the following non-linear state-dependent NDDE

$$y'(t) = \cos(t)\left(1 + y(ty(t)^2)\right) + y(t)y'(ty(t)^2) - \sin(t + tsin(t)^2)$$

with initial condition $y(0) = 0$.

Exact solution is $y(t) = \sin(t)$.

For this example, the error results of the proposed methods CLSM and DLSM using different polynomials with different values for N are presented in Tables 2 and 3.

Table 2 Error Results in CLSM for Example 2

Methods	N = 3	N = 5	N = 7
H-CLSM	2.3034e-05	2.0324e-05	3.3945e-06
B-CLSM	2.3122e-05	8.3787e-05	6.2515e-06
C-CLSM	2.3034e-05	2.5170e-05	1.6357e-05
F-CLSM	2.3034e-05	3.2888e-05	5.2660e-06

Table 3 Error Results in DLSM for Example 2

Methods	N = 3	N = 5	N = 7
H-DLSM	1.9785e-04	5.7797e-07	1.1436e-09
B-DLSM	1.9786e-04	5.7789e-07	1.3943e-09
C-DLSM	6.2817e-04	5.7731e-07	5.7731e-09
F-DLSM	1.9786e-04	5.7797e-07	1.1443e-09

4.3. Example 3

Consider the third order non-linear system of NDDE

$$y_1'''(t) = y_1'''(t-2)y_1\left(\frac{t}{3}\right) + \left(y_1(t)\right)^{\frac{2}{3}} + 2t + e^{-t}$$

$$y_2'''(t) = \frac{1}{2}y_2'''\left(\frac{t}{2}\right) + y_2'(t-1)y_1\left(\frac{t}{3}\right), t \geq 1$$

with history function $y_1(t) = e^t$ and $y_2(t) = t^2, t \in [-2, 0]$.

The given initial conditions are

$$y_1(0) = 1, y_1'(0) = 1, y_1''(0) = 1$$

$$y_2(0) = 0, y_2'(0) = 0, y_2''(0) = 2$$

The solution graphs obtained by using the proposed CLSM and DLSM with N = 7 are compared with Analytical algorithm presented in [20]. They are given in Figure 2.

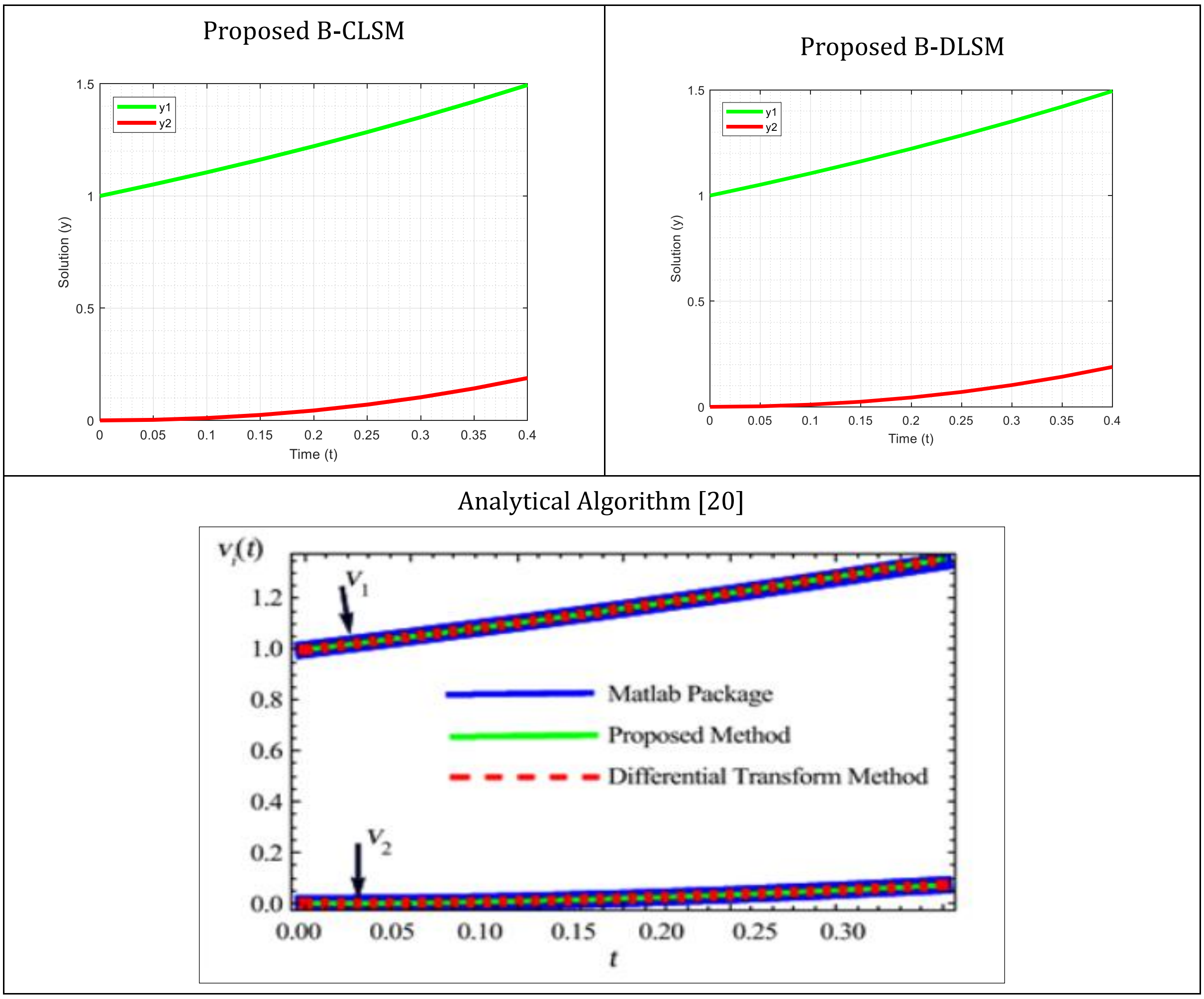


Figure 2 Comparison of Solutions for Example 3

5. Conclusion

In this study, a new approach of continuous and discrete Least square methods based on successive integration technique is proposed for solving Neutral delay differential equations. Numerical examples of linear and nonlinear NDDEs with constant, state-dependent and pantograph delays have been considered to demonstrate the efficiency of the proposed method.

The numerical results demonstrates that the proposed least square method gives results with good precision. Also, the accuracy of the results improves with increasing N (order of polynomial). Hence it is evident that the proposed method is very effective, simple, and suitable for solving linear and nonlinear NDDEs in real world problems.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

References

[1] Martsenyuk V, Klos-Witkowska A, Dzyadevych S. Sverstiuk A. Nonlinear Analytics for Electrochemical Biosensor Design Using Enzyme Aggregates and Delayed Mass Action. *Sensors.* 2022 22 (980): 1 – 17.

[2] Anusmita Das, Kaushik Dehingia, Hemanta Kumar Sarmah, Kamyar Hosseini, Khadijeh Sadri, Soheil Salahshour. Analysis of a delay-induced mathematical model of cancer. *Advances in Continuous and Discrete Models.* 2022 15 (2022): 1–20.

[3] Benito Chen-Charpentier. Delays and Exposed Populations in Infection Models. *Mathematics* 2023 11(8): 1919.

[4] Alexander Domoshnitsky, Shai Levi, Ron Hay Kappel, Elena Litsyn, Roman Yavich. Stability of neutral delay differential equations with applications in a model of human balancing. *Mathematical Modelling of Natural Phenomena.* 2021 16 (21): 1 – 18.

[5] Sami Loucif, Rafik Guefaifia, Salah Zitouni, Houssem Eddine Khochemane. Global well-posedness and exponential decay of fully dynamic and electrostatic or quasi-static piezoelectric beams subject to a neutral delay. *Zeitschrift fur angewandte Mathematik und Physik.* 2023 74 (83): 1 – 22.

[6] Z. Jackiewixa and E. Lo, The numerical solution of neutral functional differential equations by Adams predictor-corrector methods. *Applied Numerical Mathematics*. 1991 8: 477 – 491.

[7] Jafar Biazar and Behzad Ghanbari, The homotopy perturbation method for solving neutral functional-differential equations with proportional delays. *Journal of King Saud University-Science*. 2012 24 (1): 33 – 37.

[8] Xueqin Lv and Yue Gao, The RKHSM for solving neutral functional-differential equations with proportional delays. *Mathematical Methods in the Applied Sciences*, 2012 36(6): 642 – 6.

[9] M.M. Khader, Numerical and theoretical treatment for solving linear and nonlinear delay differential equations using variational iteration method. *Arab Journal of Mathematical Sciences*. 2013 19(2): 243-256.

[10] J. Ebimene Mamadu and N. Ignatius Njoseh, Solving Delay Differential Equations by Elzaki Transform Method. *Boson Journal of Modern Physics*. 2017 3(1): 214 – 219.

[11] Akmal Raza and Arshad Khand, Haar wavelet series solution for solving neutral delay differential equations, *Journal of King Saud University – Science*. 2018: 1-7.

[12] C. Dhinesh Kumar and A. Emimal Kanaga Pushpam, Higher Order Derivative Runge Kutta Method for Solving Delay Differential Equations. *Advances in Mathematics: Scientific Journal*. 2019 8(3): 26 – 34.

[13] Nur Inshirah Naziah Ismail, Zanariah Abdul Majid and Norzak Senu, Hybrid Multistep Block Method for Solving Neutral Delay Differential Equations. *Sains Malaysiana*. 2020 49: 929 – 940.

[14] J. Vinci Shaalini and A. Emimal Kanaga Pushpam, Generalized Rational Multi-step Method for Delay Differential Equations, *IAENG International Journal of Applied Mathematics*. 2020 50(1): 1 – 10.

[15] Daniele Mortari, Least-Squares solution of linear differential equations, *Mathematics*, 2017 5(48): 1-18.

[16] Siti Farhana Husin, Mustafa Mamat, Mohd Asrul Hery Ibrahim and Mohd Rivaie, Solving Ordinary Differential Equation (ODE) Using Least Square Method: Application of Steepest Descent Method, *International Journal of Recent Technology and Engineering (IJRTE)*. 2019 7(54): 524-528.

[17] Salisu Ibrahim, Numerical Approximation Method for Solving Differential Equations, *Eurasian Journal of Science & Engineering*. 2020 6(2): 157-168.

[18] Parth Singh Pawar, Dhananjay R. Mishra and Pankj Dumka, Solving First Order Ordinary Differential Equations using Least Square Method: A Comparative Study, *International Journal of Innovative Science and Research Technology*. 2022 7(3): 857-864.

[19] Salisu Ibrahim and Abdulnasir Isah, Solution for second-order Differential Equation Using Least Square Method, *Eurasian Journal of Science & Engineering.* 2022 8(1): 119 - 125.

[20] Aminu Barde and Normah Maan, Analytical Algorithm for System of Neutral Delay Differential Equations, *Applied Mathematics* 2019 10 (9): 753 – 768.