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Subdomain collocation method based on successive integration technique for solving delay differential equations

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Abstract

The main objective of this work is to propose the polynomial based Subdomain collocation method using successive integration technique for solving delay differential equations (DDEs). In this study, the most widely used classical orthogonal polynomials, namely, the Bernoulli polynomial, the Chebyshev polynomial, the Hermite polynomial, and the Fibonacci polynomial are considered. Numerical examples of linear and nonlinear DDEs have been considered to demonstrate the efficiency and accuracy of the method. Approximate solutions obtained by the proposed method are well comparable with exact solutions. From the results it is observed that the accuracy of the numerical solutions by the proposed method increases as N (order of the polynomial) increases. The proposed method is very effective, simple, and suitable for solving the linear and nonlinear DDEs in real-world problems.

Keywords: Orthogonal polynomials; Subdomain collocation method; Successive integration technique; Delay differential equations; Pantograph

1. Introduction

Delay differential equations stand as a vital class of equations. DDEs incorporate past states in the evolution of the system, where the present behavior depends not only on the current state but also on its history. DDEs plays a vital role in the fields of science and engineering. Some notable applications of DDEs are in electrodynamic model [1], economic model [2], electrochemical biosensor [3], cancer cells growth [4] and population model [5]. DDEs have been investigated by many authors and various analytical and numerical methods have been developed. Some of the numerical methods are Higher order derivative Runge Kutta method [6], Legendre pseudo spectral method [7], Sumudu transform method [8], Wavelets approach [9], Least square method based on successive integration technique [10].

The subdomain collocation method which belongs to the broader family of weighted residual methods stands out as a powerful technique for solving differential equations. This method offers a unique approach for solving complex problems by dividing the physical domain into non-overlapping subdomains and employing collocation techniques within each subdomain. Zhou *et al.* [11] proposed subdomain collocation method based on reproducing kernel approximation for solving elasticity problems. Lihua *et al.* [12] applied radial basis collocation method for fracture mechanics. Chu *et al.* [13] implemented finite subdomain collocation method with radial basis for solving singular problems. Mkhatshwa *et al.* [14] presented multi-domain multivariate spectral collocation method for solving nonlinear partial differential equations.

The above-mentioned subdomain collocation methods using different polynomials are based on operational matrices. In this study, we propose a new approach of using Subdomain collocation method based on successive integration technique for solving DDEs. This paper is organized as follows: In Section 2, the basic definitions of different polynomials

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are given. In Section 3, the description of the method for solving DDEs is presented. In Section 4, illustrative examples are provided.

2. Basic Definition of Polynomial

In this study, we consider the most widely used classical orthogonal polynomials, namely, the Hermite polynomial, the Bernoulli polynomial, the Chebyshev polynomial and the Fibonacci polynomial.

2.1. Hermite Polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty, \infty)$. There are different ways to define Hermite polynomial, one of them is the so-called Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \text{..............................}(1)$$

From Eqn. (1), the recurrence relation for the polynomial can be derived as

$$H_n(t) = 2t\, H_{n-1}(t) - H'_{n-1}(t) \text{.........................}(2)$$

$H_0(t)$ can be obtained from Eqn. (1) and the remaining terms are determined by using the recursion relation Eqn. (2). Thus, we have the following sequence of polynomial:

$$H_0(t) = 1$$

$$H_1(t) = 2t$$

$$H_2(t) = 4t^2 - 2$$

$$H_3(t) = 8t^3 - 12t$$

$$H_4(t) = 16t^4 - 48t^2 + 12$$

and so on. The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

2.2. Bernoulli Polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function for the Bernoulli polynomial of order n is defined by

$$\sum_{n=0}^{\infty} B_n(t) \frac{x^n}{n!} = \frac{xe^{xt}}{e^x - 1} \text{...................}(3)$$

The explicit formula for Bernoulli polynomial is:

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(t^k) \text{...................}(4)$$

for n ≥ 0, where B_k are the Bernoulli numbers.

$B_0(t)$ can be obtained from Eqn. (3) and the remaining terms are determined by using the recursion relation. Thus, we have few terms of the Bernoulli polynomial as:

$$B_0(t) = 1$$

$$B_1(t) = t - \frac{1}{2}$$

$$B_2(t) = t^2 - t + \frac{1}{6}$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$$

$$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$$

2.3. Chebyshev Polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1, 1]$ of order n is defined as

$$T_n(\cos t) = \cos(nt) \ldots\ldots\ldots\ldots\ldots(5)$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\, T_n(t) - T_{n-1}(t) \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

$T_0(t)$ and $T_1(t)$ can be obtained from Eqn. (5). Then the remaining terms are determined by from Eqn. (6). Thus, we have the following sequence of polynomial:

$$T_0(t) = 1$$

$$T_1(t) = t$$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 3t$$

$$T_4(t) = 8t^4 - 8t^2 + 1$$

2.4. Fibonacci Polynomial

In Mathematics, the Fibonacci polynomial is a polynomial sequence which can be considered of Fibonacci numbers. The Fibonacci polynomials are defined by a recurrence relation

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{n-1}(t) + F_{n-2}(t), & if\ n \geq 2. \end{cases}$$

The first few terms of Fibonacci polynomial are:

$$F_0(t) = 0$$

$$F_1(t) = 1$$

$$F_2(t) = t$$

$$F_3(t) = t^2 + 1$$

$$F_4(t) = t^3 + 2t$$

3. Description of the Proposed Method

Consider the n^{th} order DDE of the form

$$y^{(n)}(t) = f\big(t, y(t), y(t-\tau), y'(t), y'(t-\tau),), \ldots, y^{(n-1)}(t), y^{(n-1)}(t-\tau)\big), t > t_0 \ (7)$$

with initial conditions

$$y^{(i)}(t_0) = \emptyset(t), i = 1, 2, 3, \ldots\ t \leq t_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Here $\emptyset(t)$ is the initial function and τ is the delay term.

Let P(t) represent any orthogonal polynomial. For the proposed method, we assume that

$$y^{(n)}(t) \approx B^T P(t)^T = \sum_{j=0}^{N} c_j P_j(t) \quad \text{……………………. (9)}$$

where N being any positive integer.

$$B^T = (c_0, c_1, \dots c_N)$$

$$P(t) = \big(P_0(t), P_1(t) \dots P_N(t)\big)$$

Our aim is to determine the polynomial coefficients $c_j's$. For this, we integrate Eqn. (9) with respect to t from t_0 to t,

$$\left.\begin{aligned} y^{(n-1)}(t) &= y(t_0) + \int_{t_0}^{t} B^T\, P_j(t)\, dt \\ y^{(n-2)}(t) &= y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T\, P_j(t)\, dt \\ &\cdots \\ y'(t) &= \sum_{i=0}^{n-1} y^{(i)}(t_0) + \underbrace{\int_{t_0}^{t}\int_{t_0}^{t}\dots\int_{t_0}^{t}}_{(n-1)\ times} B^T\, P_j(t)\, dt \\ y(t) &= \sum_{i=0}^{n} y^{(i)}(t_0) + \underbrace{\int_{t_0}^{t}\int_{t_0}^{t}\dots\int_{t_0}^{t}}_{n\ times} B^T\, P_j(t)\, dt \end{aligned}\right\} \text{… … … … … …. (10)}$$

Now, for delay terms

$$\left.\begin{aligned} y^{(n-1)}(t-\tau) &= y(t_0) + \int_{t_0}^{t} B^T\, P_j(t-\tau)\, dt \\ y^{(n-2)}(t-\tau) &= y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T\, P_j(t-\tau)\, dt \\ &\cdots \\ y'(t-\tau) &= \sum_{i=0}^{n-1} y^{(i)}(t_0) + \underbrace{\int_{t_0}^{t}\int_{t_0}^{t}\dots\int_{t_0}^{t}}_{(n-1)\ times} B^T\, P_j(t-\tau)\, dt \\ y(t-\tau) &= \sum_{i=0}^{n} y^{(i)}(t_0) + \underbrace{\int_{t_0}^{t}\int_{t_0}^{t}\dots\int_{t_0}^{t}}_{n\ times} B^T\, P_j(t-\tau)\, dt \end{aligned}\right\} \text{… … … …. (11)}$$

Then we substitute (10) and (11) in (7) to get the residue function $R(t)$. In the subdomain method, the physical domain is divided into non-overlapping N subdomains. Here N is taken as the number of polynomial coefficients. Each weight function is selected as unity in the respective subdomain and zero in the remaining parts. That is,

$$w_i = \begin{cases} 1, & if\ t_i \le t \le t_{i+1} \\ 0, & else \end{cases} \quad (i = 1,2, \dots, N) \quad \text{……………………… (12)}$$

Then, the average residual over each subdomain is made to zero.

$$\int_a^b w_i\, R(t)\, dt = \int_{t_i}^{t_{i+1}} R(t)\, dt = 0, (i = 1,2, \dots, N) \quad \text{………………. (13)}$$

This yields a system of linear or nonlinear algebraic equations subject to the linear and nonlinear terms in Eqn. (7). On solving this system of algebraic equations, we get the respective polynomial coefficients c_j's from which the solution of the DDE (7) can be obtained.

4. Numerical Simulations

In this section, three numerical examples are given to demonstrate the accuracy and effectiveness of the proposed method. The proposed method based on successive integration technique uses Hermite, Chebyshev, Bernoulli and Fibonacci polynomials for solving linear and nonlinear DDEs.

4.1. Example 1

Consider the third order linear delay differential equation with constant delay

$$\frac{d^3y(t)}{dt^3} = -y(t) - y(t-0.3) + e^{-t+0.3}, 0 \leq t \leq 1$$

with initial conditions

$$y(0) = 1, y'(0) = -1 \text{ and } y''(0) = 1.$$

The exact solution is $y(t) = e^{-t}$.

The numerical results are obtained by the subdomain method using different polynomials with various values of N. The absolute errors are presented in Tables 1 and 2. The solution graph for N = 7 using Hermite polynomial is presented in Fig. 1.

Table 1 Absolute Errors at t = 1 for Example 1

Polynomials	**N = 3**	**N = 5**	**N = 7**
Hermite	2.16 e-06	4.72 e-09	1.12 e-11
Bernoulli	2.16 e-06	4.72 e-09	1.12 e-11
Chebyshev	2.16 e-06	4.72 e-09	1.12 e-11
Fibonacci	2.16 e-06	4.72 e-09	1.12 e-11

Table 2 Absolute Errors in Example 1 (N = 7)

t	**Hermite**	**Bernoulli**	**Chebyshev**	**Fibonacci**
0.2	8.99 e-13	8.99 e-13	8.99 e-13	8.99 e-13
0.4	2.56 e-12	2.56 e-12	2.56 e-12	2.56 e-12
0.6	4.89 e-12	4.89 e-12	4.89 e-12	4.89 e-12
0.8	7.85 e-12	7.85 e-12	7.85 e-12	7.85 e-12
1.0	1.12 e-11	1.12 e-11	1.12 e-11	1.12 e-11

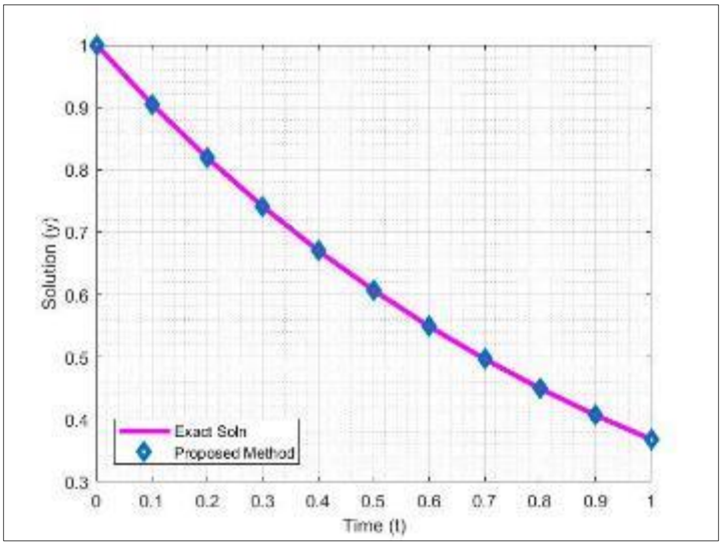

Figure 1 Solution graph for Example 1 using Hermite polynomial

4.2. Example 2

Let us consider the following nonlinear DDE with variable delay

$$y'(t) + ty(t - t^2) + ty^2 = 1 + t^2, 0 \leq t \leq 1$$

with the initial condition

$$y(0) = 0, t \epsilon [0,1].$$

The analytical solution is $y(t) = t$.

The numerical results are obtained by the subdomain method using different polynomials with various values of N. The absolute errors are presented in Table 3. The solution graph for N = 7 using Bernoulli polynomial is presented in Fig. 2.

Table 3 Absolute Errors in Example 2

Polynomials	N = 3	N = 5	N = 7
Hermite	5.16 e-03	1.19 e-06	4.03 e-14
Bernoulli	5.16 e-03	1.19 e-06	4.03 e-14
Chebyshev	5.16 e-03	1.19 e-06	4.03 e-14
Fibonacci	5.16 e-03	1.19 e-06	4.03 e-14

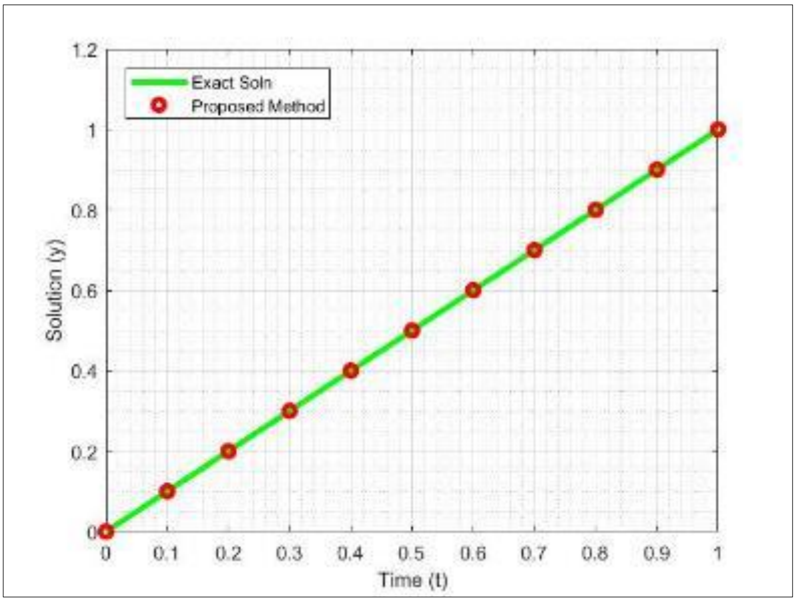

Figure 2 Solution graph for Example 2 using Bernoulli polynomial

4.3. Example 3 – Parkinson's Disease Model [15]

$$y'(t) = 0.1y(t-2) + 0.2y(t-5) + 0.3y(t-2)y(t-3), 0 \leq t \leq 10$$

with history function as

$$y(t) = 0.5, t > 0 \text{ and } -3 < t < 0.$$

For this example, the numerical results are obtained by using the proposed method based on the different polynomials. The numerical simulations by the proposed methods are compared with the simulation by Step method using Picard approximation [15]. These have been shown in Fig. 3 and Fig. 4.

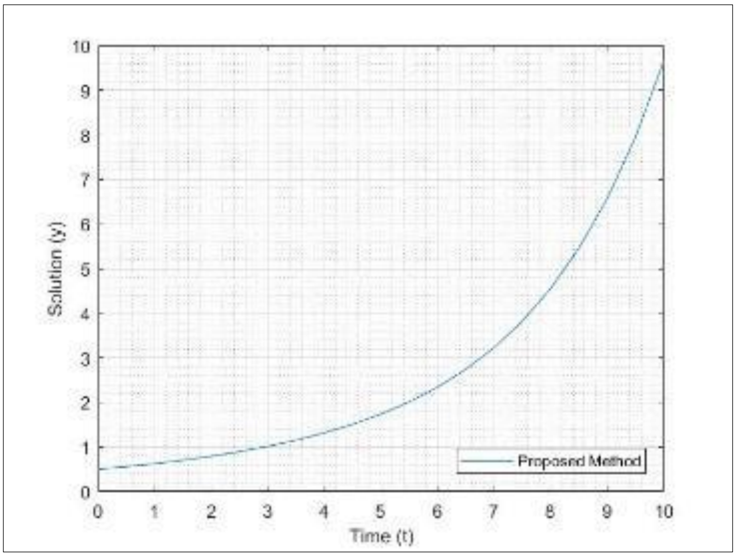

Figure 3 Solution graph for Example 3 using Hermite polynomial

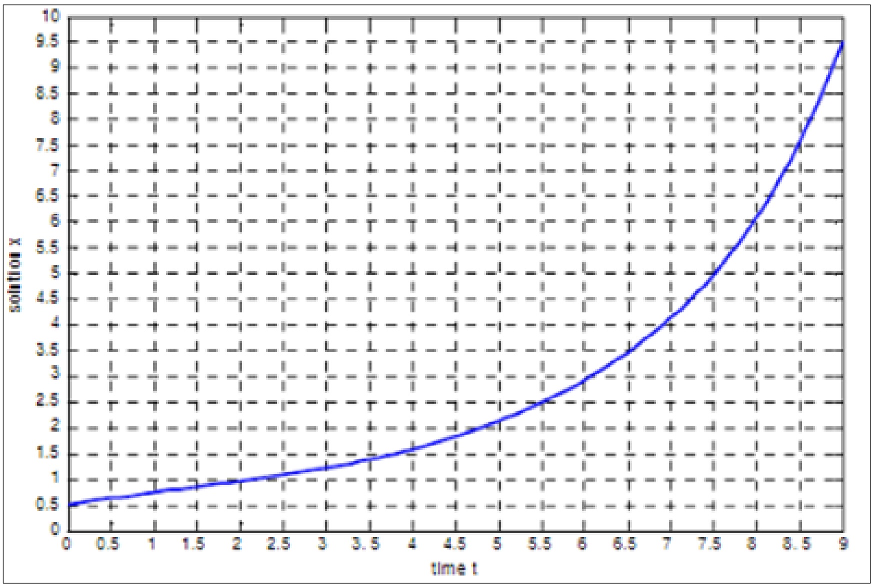

Figure 4 Numerical Simulations by Step-Method [15]

5. Conclusion

In this paper, a new approach of subdomain collocation method based on successive integration technique is proposed for solving delay differential equations. Numerical examples of linear and nonlinear DDEs with constant, variable and pantograph delays have been considered to demonstrate the efficiency of the proposed method.

The numerical results demonstrates that the proposed method gives results with good precision. Also, the accuracy of the results improves with increasing N (order of polynomial). Hence it is evident that the proposed method is very effective, simple, and suitable for solving linear and nonlinear DDEs in real world problems.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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